THE PROBABILITY OF THE ARITHMETIC MEAN COMPARED WITH THAT OF CERTAIN OTHER FUNCTIONS OF THE MEASUREMENTS.

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1. Introduction.

In his Theorie der Beobachtungsfehler, Czuber has exhibited many of the attempts made to unite the principle of the arithmetic mean as the "most probable value" with the Gaussian probability law. He* quotes from Bertrand† who gives an example to show that this law and principle are not strictly compatible. It is one object of this paper to show this incompatibility by other methods,—to exhibit functions of the measurements to which the Gaussian law assigns a greater probability than it assigns to the arithmetic mean.‡

Wrapt up with the Gaussian law are several assumptions. Of these, we note the following in particular.

1. A true value, a, exists for the unknown.§

2. Associated with a measurement, m, or with a set of n measurements taken under similar circumstances, there exists a positive constant, h, called the measure‖ of precision.

3. An objective¶ or physical probability may exist, when its value is unknown or but "approximately" known.

For example: if an urn contains just n balls of which w are white, the

* Loc. cit., p. 51.

† Calcul des Probabilités (1889), p. 180.

‡ Sets of axioms have been proposed to ground the principle of the arithmetic mean as the "most probable value." See Czuber, loc. cit., p. 16–47; also Schimmack, Mathematische Annalen, 68. Band, p. 125. These axioms are, in general, of such a nature that they may be used equally well to ground the principle that the arithmetic mean is the least probable value.

§ This assumption and the next one presuppose that a unit of measure has been adopted. In changing from meters to centimeters, a would be multiplied by 100, h divided by 100, but ha would be invariant.

‖ Here we merely set up the "assumption" or "axiom" that h exists. A commonly accepted approximation for h is $\sqrt{\frac{n-1}{2\Sigma v^2}}$, in which Σv^2 is the sum of the squares of the residuals of the measurements; i. e., $v_i = M - m_i$, M being the arithmetic mean. See Czuber, Wahrscheinlichkeitsrechnung, I, p. 281.

¶For the view that probability is "purely subjective," see Sigwart's Logic, trans. by Dendy, vol. 2, p. 224. For a distinction between objective probability and subjective probability, see Kries, "Die Prinzipien der Wahrscheinlichkeitsrechnung (1836), p. 95. Reference is made to the distinction in the Encyklopädie (I, D. 1), p. 735.

probability of its delivering a white ball is w/n,—whether anyone knows what n and w are, or not. Note that h in 2. is unknown, as well as a in 1.

The Gaussian probability law then states that

$$P = \frac{h}{\sqrt{\pi}} \int_{x'}^{x''} e^{-h^2x^2} dx \tag{1}$$

is the probability that the error, $x = a - m$, will lie between x' and x'', where $x' \leqq x''$. The following, also, are important assumptions underlying (1), or inferences from (1), according to the viewpoint.

4. The probability, P, is a function of h, x', and x'', but not of a.*

This does not prevent the probability of the error of certain functions of the measurements being functions of a, h, x' and x''.

5. P is never zero when x' is less than x''. Roughly speaking: "Any error is possible."

6. P is zero if x' equals x''. "The probability of any particular error is zero."

7. P is unchanged, if $-x''$, $-x'$ replace x', x''. "The probability of a negative error is the same as that of the corresponding positive error."

8. P is greater for the interval $(-\alpha, +\alpha)$ than for any other interval of length, 2α. "Zero is the most probable error."

9. P is unity for the interval $(-\infty, +\infty)$. Thus the probability for very large errors is very small.

These assumptions, in general, are mathematical refinements, or abstractions,—somewhat comparable with the conception in geometry of a line with no breadth or thickness. However, it is not the object of this paper to defend these assumptions or to prove the Gaussian law, but to investigate its consequences.

From 6., it follows that under the Gaussian law, there is, strictly speaking, no most probable error. Each of the infinite number (see 5.) of possible errors has the same probability, zero.† For the discussion of the relative magnitude of probabilities, some definition is needed. Corresponding to each function, f, of the measurements, there is an error, $a - f$. When f lies in the interval from $a - \alpha$ to $a + \alpha$, its error lies in the interval from $-\alpha$ to α.

Definition.—The probability of f_1 will be said to be greater than that of f_2, if the probability that the error of f_1 will‡ lie in the interval from $-\alpha$ to $+\alpha$

* See Bertrand, loc. cit., p. 177.

† Similarly, the probability, a posteriori, that any particular real number be the true value is zero,—according to standard treatments. For example, see Poincaré, Calcul des Probabilités (1896), p. 149. Set $dz = 0$ in the numerator. There is here, then, no "most probable value" for the unknown true value.

‡ The future tense. By "measurements," as discussed here, will be meant contemplated measurements.

is greater than the probability that the error of f_2 *will lie in the same interval, for all positive values of* α *less than some* α',*—in other words, if the probability that* f_1 *will differ from a by less than* α *is greater than that* f_2 *will differ from a by less than* α*, when* $\alpha < \alpha'$.

Some such restriction as that imposed upon α is needed to avoid anomalies. For example: if α is taken equal to a, the probability of bm is greater than that of m, where b is any proper fraction. For bm will lie in the interval, $(a - \alpha,\ a + \alpha) = (0,\ 2a)$, whenever m lies in $(0,\ 2a/b)$. Furthermore, it is natural to make α small; for, in general, the error of each measurement will be small in comparison with a. But α cannot be made zero; for the probability that f_1 or f_2 will equal a is zero.* For example: the probability that the average of four measurements will differ from a by less than α, is found by replacing h by $2h$, and x', x'', by $-\alpha$, α in (1). This probability vanishes with α.

2. The Comparative Probabilities of the Arithmetic Mean of Measurements and the Square Root of the Mean of Their Squares.

Under the Gaussian law (1), it follows that the probability, P_α, that the average, M, of two measurements will lie in the interval, $(a-\alpha,\ a+\alpha)$, is given by

$$P_\alpha = \frac{2\sqrt{2}h}{\sqrt{\pi}}\int_0^\alpha e^{-2h^2x^2}dx = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{2}h\alpha} e^{-t^2}dt = \Theta(\sqrt{2}h\alpha). \tag{2}$$

This is not a new theorem. It may be proved as follows.† Let the error of m_1 be x, and that of m_2 be y. P_α is then the probability that

$$-\alpha \leqq a - \frac{m_1 + m_2}{2} \leqq \alpha,$$

or that

$$\begin{aligned} -2\alpha &\leqq x + y \leqq 2\alpha, \\ -2\alpha - x &\leqq y \leqq 2\alpha - x. \end{aligned} \tag{3}$$

The first error, x, may be of any magnitude; but the second, y, must then satisfy (3). Hence,

$$P_\alpha = \frac{h}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-h^2x^2}dx\, \frac{h}{\sqrt{\pi}}\int_{-2\alpha-x}^{2\alpha-x} e^{-h^2y^2}dy.$$

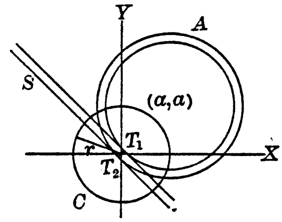
* Except for some trivial function, as $0m + a$.

† The proof given here is regarded as simpler than a proof using a non-convergent "discontinuity factor," such as

$$\int_{-\infty}^{\infty}\cos(\kappa\theta)d\theta \quad \text{or} \quad \int_{-\infty}^{\infty} e^{\kappa\theta\sqrt{-1}}\,d\theta.$$

Likewise, it may be proved that the probability that the error, E, of M will lie in $(0,\ \alpha)$ is $\frac{1}{2}\Theta(\sqrt{2}h\alpha)$. Thus the probability that E will lie in $(x',\ x'')$ is given by (1) with h changed to $\sqrt{2}h$.

The field of integration, S, is bounded by two parallel lines, and its width is $2\sqrt{2}\alpha$. If the axes are rotated through 45°, the integrand is unchanged, but the boundaries of the field become parallel to the Y axis, at a distance of $\sqrt{2}\alpha$ from it. This gives (2).



Now let

$$\sqrt{\frac{m_1^2 + m_2^2 + \cdots + m_n^2}{n}}$$

be called the root-mean-square of the n measurements.

THEOREM 1. *Under the Gaussian probability law, the root-mean-square of two measurements has a greater probability than their arithmetic mean, provided the product of the precision constant by the true value is greater than* 2.

Proof.—The condition mentioned is

$$ha > 2. \tag{4}$$

Now h is positive, and thus a is also.

Let P_α' be the probability that

$$a - \alpha \leqq \sqrt{\frac{m_1^2 + m_2^2}{2}} \leqq a + \alpha.$$

The errors being x and y as before, x and y must satisfy

$$2(a - \alpha)^2 \leqq (x - a)^2 + (y - a)^2 \leqq 2(a + \alpha)^2; \tag{5}$$

in this, α is to be taken less than a. The point, (x, y), is then confined to an annular region, A, bounded by two concentric circles, with centers at (a, a) and radii, $\sqrt{2}(a - \alpha)$ and $\sqrt{2}(a + \alpha)$, respectively. The width of the ring is $2\sqrt{2}\alpha$.

Then

$$P_\alpha' = \frac{h^2}{\pi}\int_A\int e^{-h^2(x^2+y^2)}dxdy, \tag{6}$$

whereas P_α is a like integral taken over the strip, S, of the same width, $2\sqrt{2}\alpha$. If the integrand is set equal to z, the locus is a surface of revolution about the Z axis, and thus the integrand takes the same value for every point on a circle, C, of radius, r, in the XY plane, centered at the origin. Consider now the evaluation of P_α' and P_α, under a transformation to polar coördinates. If

$$\sqrt{2}\alpha < r < \sqrt{2}(2a - \alpha),$$

arcs of the circle, C, will be intercepted between the boundaries of S and also of A; but the latter arcs will be the greater, because their chords are greater,—a straight line segment joining a point on one boundary of A

to a point on the other boundary will be greater than $2\sqrt{2}\alpha$, unless it is a portion of the radius of the outer circle. Now, usually the integrand in (6) will become inappreciable long before r reaches $2\sqrt{2}a$; usually, errors double the true value are well-nigh impossible. The infinite portions of S contribute next to nothing—when $ha > 2$—to the integral, P_a; whereas P_a' outstrips P_a in the stretch from 0 to $2\sqrt{2}a$. To get a numerical relation between P_a and P_a', we may proceed as follows:

Let the axes be rotated through 45°; and set $R = a\sqrt{2}$, $\delta = \alpha\sqrt{2}$. Let the intersections of C with the boundaries of A in the first quadrant be (x_1, y_1) and (x_2, y_2) when

$$\delta < r < 2R - \delta. \tag{7}$$

Find the value of $y_2 - y_1$ and to it apply the inequality,

$$\sqrt{c+d} - \sqrt{c-d} > d/\sqrt{c}, \qquad \text{if } 0 < d < c.$$

Then, if D is the length of the chord,

$$D^2 > \frac{4\delta^2}{1 - \dfrac{r^2 - \delta^2}{4R^2}},$$

$$D > 2\delta\left[1 + \frac{r^2 - \delta^2}{8R^2}\right].$$

Then if θ is the angle formed by straight lines from the origin to (x_1, y_1) and (x_2, y_2), respectively,

$$\theta > \frac{2\delta}{r}\left[1 + \frac{r^2 - \delta^2}{8R^2}\right]. \tag{8}$$

Now, if (7) did not need to be satisfied, it would be found upon passing to polar coördinates, and using

$$\frac{2h}{\sqrt{\pi}}\int_0^\infty e^{-h^2r^2}dr = 1, \quad \frac{2h}{\sqrt{\pi}}\int_0^\infty e^{-h^2r^2}r^2dr = \frac{1}{2h^2},$$

that P_a' would—when δ is small—be greater than

$$\frac{2\delta h}{\sqrt{\pi}}\left[1 + \frac{1}{32h^2a^2}\right]. \tag{9}$$

But, because (8) was obtained from (7) with $r < 2R - \delta$, a deduction must be made from the bracket in (9) of an amount not greater than $3/[80(ha)^3]$, together with an amount which approaches zero with δ,—

because of the inequality, $\delta < r$, in (7). The upper limit mentioned for the former deduction is obtained by using the inequalities,

$$\int_{2R}^{\infty} e^{-h^2r^2}dr = \frac{1}{h}\int_{2hR}^{\infty} e^{-t^2}dt < \frac{1}{2h^2R}\int_{2hR}^{\infty} e^{-t^2}tdt,$$

$$\int_{2R}^{\infty} e^{-h^2r^2}r^2dr < \frac{1}{2R}\int_{2R}^{\infty} e^{-h^2r^2}r^3dr.$$

But

$$P_a < \frac{2\delta h}{\sqrt{\pi}};$$

and hence if $ha > 2$, and δ sufficiently small,

$$P_a' > P_a + \frac{2\delta h}{\sqrt{\pi}}\left[\frac{1}{80(ha)^2}\right]. \qquad \text{Q.E.D.}$$

It is not to be supposed that 2 is the critical value for ha. But it is evident from (6) that, for a given a, it would be possible to choose an h so small that the integrand in (6) would be sensibly unity throughout A and the nearer portions of S. The remote portions of S would then make P_a greater than P_a'.

THEOREM 2. *Under the Gaussian law, the probability of the arithmetic mean of three measurements is greater than the probability of their root-mean-square.*

Proof.—In this case, the probability, P_a, that the arithmetic mean will differ from the true value by less than α, is $\Theta(\sqrt{3}h\alpha)$. It may be found as a triple integral over a region bounded by parallel planes, each at a distance, $\sqrt{3}\alpha$, from the origin,—analogous to S in the figure. Likewise, for three measurements, a being positive, P_a' is the integral over a region between two concentric spheres, centered at (a, a, a), and tangent to the two planes bounding the new S. The zones of the sphere, C, cut off by the regions, S and A, have the same altitude and thus the same area. On these zones —for fixed r—the integrand is a constant, and each of the two zones gives the same integral. But as r goes from zero toward infinity, the region A becomes exhausted, whereas S does not. Hence $P_a' < P_a$. But, in general, their difference would be inappreciable. No condition, such as $a > 0$, is needed in dealing with three measurements. For, in case $a = 0$, the region A becomes a sphere lying in S. And if $a < 0$, P_a' is zero for a small α,—if the usual convention, giving to the radical the positive sign, is adopted. But if the radical is to be made always negative, the treatment is essentially that for $a > 0$.

When n is very large, a certain presumption exists that the arithmetic

mean, M, has a greater probability than the root-mean-square, M'. For if M is positive, and $v_1, v_2, \cdots$ are the residuals, then

$$M' = M\left[1 + \frac{\Sigma v^2}{nM^2}\right]^{\frac{1}{2}}.$$

Now the arithmetic mean is subject to the Gaussian law with precision constant, $\sqrt{n}h$. And it will be proved presently that the probability of bM is less than that of M, when b is a constant greater than unity. When n is very large, the bracket above is supposed to be sensibly a constant, and it is greater than unity.

That the arithmetic mean, M, of n measurements,—each subject to the Gaussian law with precision, h,—is subject to this law, with precision $\sqrt{n}h$, may be proved as follows. The condition that the error of M shall lie in $(-\alpha, \alpha)$, is equivalent to the condition that the sum, Σx, of the errors of the measurements shall lie in $(-n\alpha, n\alpha)$. The probability for this is an n-fold integral taken over a region bounded by two "parallel planes" in n dimensions. Each plane is at a "distance," $\sqrt{n}\alpha$ from the origin. By an orthogonal transformation—"rotation"—the planes can be made "perpendicular" to an "axis." The following is such a transformation:

$$X_1 = \frac{x_1}{\sqrt{n}} + \frac{x_2}{\sqrt{n}} + \frac{x_3}{\sqrt{n}} + \cdots + \frac{x_n}{\sqrt{n}},$$

$$X_2 = \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}},$$

$$X_3 = \frac{x_1}{\sqrt{2 \cdot 3}} + \frac{x_2}{\sqrt{2 \cdot 3}} - \frac{2x_3}{\sqrt{2 \cdot 3}},$$

$$X_r = \frac{x_1}{\sqrt{(r-1)r}} + \frac{x_2}{\sqrt{(r-1)r}} + \cdots + \frac{x_{r-1}}{\sqrt{(r-1)r}} - \frac{(r-1)x_r}{\sqrt{(r-1)r}},$$

where $2 \leqq r \leqq n$. Then $X_1 = \Sigma x/\sqrt{n}$, and lies in $(-\sqrt{n}\alpha, \sqrt{n}\alpha)$.

3. The Comparative Probabilities of m and bm, and also of M and bM.

Since the average, M, of n measurements, is subject to the Gaussian probability law, with precision constant, $\sqrt{n}h$, a comparison of the probabilities of m and bm—where b is a constant—is likewise a comparison of the probabilities of M and bM,—with the proper change from h to $\sqrt{n}h$.

THEOREM 3. *Under the Gaussian law, the probability of bm is less than that of m, if b is a constant greater than unity; but there exist positive values of the constant b for which the probability of bm is greater than that of m.*

Proof.—Let P_a be the probability that m will differ from a by less than α, and let P_a' be the probability that

$$a - \alpha \leqq bm \leqq a + \alpha,$$

or

$$-\frac{a+\alpha}{b} \leqq -m \leqq -\frac{a-\alpha}{b},$$

$$a - \frac{a+\alpha}{b} \leqq a - m \leqq a - \frac{a-\alpha}{b}.$$

By hypothesis, this probability is

$$P_a' = \frac{h}{\sqrt{\pi}} \int_{x'}^{x''} e^{-h^2x^2}\, dx, \tag{10}$$

where

$$x' = a - \frac{a+\alpha}{b}, \qquad x'' = a - \frac{a-\alpha}{b}.$$

The interval of integration is, in length, $2\alpha/b$. In the special case, where $a = 0$, $P_a' > P_a$ if $0 < b < 1$; but $P_a' < P_a$ if $b > 1$. Also, in all other cases,

$$P_a' < P_a, \qquad \text{if } b > 1.$$

For the interval of integration for P_a' is smaller, and is less favorably situated—being centered at $a - a/b$. Likewise, when $0 < b < 1$, the center of the interval is at $a - a/b$; but the length of the interval is greater than 2α, the interval for P_a. It will now be shown that if b is taken sufficiently close to unity—thus bringing the center of the interval close to the origin—the advantage which P_a' has in length of interval, outweighs the disadvantage in position; i. e., $P_a' > P_a$.

Now the integrand is greatest when $x = 0$. Hence

$$P_a < \frac{2h\alpha}{\sqrt{\pi}}.$$

On the other hand, if $a > 0$, the integrand for P_a' is least when $x = a - (a + \alpha)/b$. Thus

$$P_a' > \frac{2\alpha}{b}\frac{h}{\sqrt{\pi}} e^{-h^2[a(1-b)+\alpha]^2/b^2}.$$

Hence $P_a' > P_a$, provided b and α can be so taken that

$$\frac{1}{b} e^{-h^2[a(1-b)+\alpha]^2/b^2} > 1,$$

that is,

$$\frac{h^2}{b^2}[a(1-b)+\alpha]^2 < \log_e \frac{1}{b}.$$

Take $\alpha \leqq a(1 - b)$, and let $1/b = 1 + y$. It is then to be shown that y can be taken positive but small enough so that

$$4h^2a^2y^2 < \log_e (1 + y).$$

But when y is sufficiently small,

$$4h^2a^2y^2 < y - \frac{y^2}{2} < \log_e (1 + y). \qquad \text{Q.E.D.}$$

The proof for the case, $a < 0$, is essentially the same.

Example.—If the Gaussian law be assumed, and if it be admitted that of two values for the unknown it is better to accept that which has the greater probability, in accordance with the definition adopted in this paper, then the foregoing theorem implies that if a meter bar is measured in inches with the result, 39.37, it is better to accept for the length of the bar some number a little less than 39.37 than to accept 39.37 itself. This applies whether 39.37 is a single measurement or is the arithmetic mean of a set of measurements taken under the same circumstances,—for the arithmetic mean is subject to the Gaussian law when the individual measurements are thus subject.

But the difference of 39.37 and such a number is so small—if the measurements have been made with even a moderate degree of accuracy—as to be negligible. This may be seen as follows. With α small, the integral (10), as a function of b, has a maximum for approximately

$$b = 1 - \frac{1}{2h^2a^2}. \tag{11}$$

Furthermore, $P_\alpha' > P_\alpha$ if

$$1 > b > 1 - \frac{2}{2h^2a^2 + 3}, \tag{12}$$

and α is small enough. Or, if the arithmetic mean of n measurements is used, then in place of (12), we have

$$1 > b > 1 - \frac{2}{2nh^2a^2 + 3}. \tag{13}$$

But, ordinarily, h^2a^2 is very large, and this multiplier, b, to be used upon m or upon the arithmetic mean, M, does not differ appreciably from unity. Indeed, in the case of the meter bar, it may be reasonably certain that $a > 39$. Now the commonly accepted formula* connecting h with the so-called "probable error," r, is

$$hr = .476936.$$

* Czuber, Wahrscheinlichkeitsrechnung, I, p. 270.

To take $r = .01$ signifies that we suppose it as likely that the error of a measurement will be numerically less than a hundredth of an inch as that it will exceed that amount. In this case, $h > 47$; and, hence,

$$h^2a^2 > 3,000,000.$$

A modification of the foregoing theorem consists in the use of a function, $bm + c$, with $b < 1$ and $c > 0$,—provided $a > 0$. The c does not affect the length of the interval of integration, but merely its position. With b fixed, there exist positive values of c which move the interval toward the origin and thus augment the probability.

THEOREM 4. *Under the Gaussian probability law, the probability of $m + c$ is less than that of m, for every value of the constant, c, not zero.*

The proof of this theorem follows at once from considerations given above.

4. The Probability of the Median.

By the median of $2\nu + 1$ measurements is meant the middle or $(\nu+1)$th measurement when they are arranged according to magnitude. If there are three measurements, the first or second or third measurement made —in order of time—may be the median. If the first is the median, the measurement less than the median may be the second, or it may be the third. The probability that the error of m_1 will lie in $(-\alpha, \alpha)$ is very nearly $2h\alpha/\sqrt{\pi}$ when α is small; the probability that m_2 will then be less than m_1 is nearly $1/2$; and the probability that m_3 will then be greater than m_1 is nearly $1/2$. There being six arrangements of the m's, the probability of the median is nearly $3h\alpha/\sqrt{\pi}$. This is about 87 per cent of the approximate probability, $2\sqrt{3}h\alpha/\sqrt{\pi}$, of the arithmetic mean. The exact probability, —according to the Gaussian law—that the median of $2\nu + 1$ measurements will lie in $(a - \alpha, a + \alpha)$ is

$$P_a' = \frac{(2\nu + 1)!}{4^\nu(\nu!)^2} \frac{h}{\sqrt{\pi}} \int_{-a}^{a} [1 - \Theta^2(hx)]^\nu e^{-h^2x^2} dx. \tag{14}$$

In this expression, Θ and thus its square become negligible when, for any given ν, α is made sufficiently small.

By Stirling's formula,*

$$1 \cdot 2 \cdot 3 \cdots \nu = \nu! = \sqrt{2\pi\nu}\, \nu^\nu e^{-\nu+\theta/12\nu}, \qquad 0 < \theta < 1.$$

Hence, if P_a is the probability that the arithmetic mean will lie in $(a - \alpha, a + \alpha)$,

* See, for example, Broggi, Traité des Assurances sur la Vie (1907), p. 54.

$$\frac{P_\alpha'}{P_\alpha} = \sqrt{\frac{2\nu + 1}{\pi\nu}}\, e^{\theta/24\nu - \theta'/6\nu}(1 + \epsilon),$$

where $\lim_{\alpha=0} \epsilon = 0,\ 0 < \theta < 1,\ 0 < \theta' < 1.$

THEOREM 5. *The probability of the median of an odd number,* $2\nu + 1$, *of measurements is less than that of their arithmetic mean—under the Gaussian law—and if* ν *is made sufficiently large, and then* α *taken small enough, the ratio of* P_α' *to* P_α *can be made as near* $\sqrt{2}/\sqrt{\pi} = .7979$ *as we please.*

Thus, with a large number of measurements, the probability of the median falls about 20 per cent short* of that of the arithmetic mean.

5. The Probability of the Geometric Mean.

By the geometric mean of two positive measurements, m_1 and m_2, will be meant $+\sqrt{m_1 m_2}$; but the negative radical will be taken if both are negative; and the mean will not be regarded as defined, if they have unlike signs. Likewise, for n measurements, the geometric mean will be regarded as positive, if all measurements are positive; negative, if all measurements are negative; otherwise, undefined.

THEOREM 6. *The probability of the geometric mean is less than that of the arithmetic mean, under the Gaussian law.*

This can be proved for the three cases: $a > 0$, $a = 0$, $a < 0$. In the case of two measurements, when $a > 0$, the field of integration for the geometric mean is bounded by two equilateral hyperbolas, tangent to the boundaries of S—see figure—at T_1 and T_2; and extending out the second quadrant, and down the fourth quadrant, asymptotic to the lines, $y = a$, and $x = a$. The segment, T_1T_2, is the only straight line segment with slope, unity, joining the two hyperbolas, and having a length as great as $2\sqrt{2}\alpha$. If the axes are rotated through 45°, and then integration is performed first with respect to x; the integral with y fixed and not zero—will be less for the geometric mean than for the arithmetic mean. In n dimensions, the proof is facilitated, by translating to $(a, a \cdots a)$ as a new origin, using the "surfaces" upon which the points have positive coordinates, and showing that if $(x_1, x_2, \cdots x_n)$ lies on one surface, $(x_1 + 2\alpha, x_2 + 2\alpha, \cdots x_n + 2\alpha)$ falls "outside" the region bounded by the two surfaces.

6. The Probability of the Weighted Mean.

By a weighted mean of n measurements is meant,

* This does not necessarily discredit the use of the median in economic, biological or other investigations. Only Gaussian distributions are being considered in this article.

$$p_1m_1 + p_2m_2 + \cdots + p_nm_n,$$

where the p's are given constants such that

$$p_1 + p_2 + \cdots + p_n = 1.$$

THEOREM 7. *If n measurements are subject to the Gaussian law, with precision constants, h_1, h_2, $\cdots$ h_n, respectively, then any weighted mean is also subject to the Gaussian law,* with precision constant, H, where*

$$\frac{1}{H^2} = \Sigma\left(\frac{p}{h}\right)^2.$$

This H takes its greatest value when

$$p_1 = \frac{h_1^2}{\Sigma h^2}, \quad p_2 = \frac{h_2^2}{\Sigma h^2}, \quad \cdots, \quad p_n = \frac{h_n^2}{\Sigma h^2};$$

and in this case,

$$H^2 = \Sigma h^2;$$

and the probability of this weighted mean is greater than that of any other weighted mean.

The proof of this theorem † involves the use of an orthogonal transformation which can be set up with zero coefficients in the same places as in the orthogonal transformation indicated for the arithmetic mean. Now H has a maximum when $\Sigma(p/h)^2$ has a minimum. The case, $p_n = 0$, can be considered separately. Otherwise,

$$p_n = 1 - p_1 - p_2 - \cdots - p_{n-1}$$

can be inserted, and the minimum located by setting the first partial derivatives equal to zero. For this point, an actual minimum and least value will occur; since

$$\Delta\Sigma\left(\frac{p}{h}\right)^2 = \sum_1^{n-1}\left(\frac{\Delta p}{h}\right)^2 + \frac{1}{h_n^2}\left(\sum_1^{n-1}\Delta p\right)^2.$$

It should be noted that this weighted mean has not been proved to be the "most probable value." In fact, since this weighted mean, W, is subject to the Gaussian law, there are constants, b, a little less than unity, —as has been proved in Theorem 3—such that bW has a greater probability than W.

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* Cf. Czuber, Wahrscheinlichkeitsrechnung, I, p. 260.

† For a generalization of this theorem, see: Dodd, "The Least Square Method grounded with the aid of an Orthogonal Transformation," Jahresbericht der Deutschen Mathematiker-Vereinigung, 21. Band (1912), p. 177.

Corollary.—The probability of the arithmetic mean of n measurements with the same precision, h, is greater than the probability of any other linear homogeneous function of the measurements with constant coefficients whose sum is unity.

For, in the first place, the arithmetic mean is the weighted mean for which each weight is $1/n$. This weight, $1/n$, is the most favorable weight for each measurement, when all the h's are equal. The general formula,

$$H^2 = \Sigma h^2,$$

becomes in this case,

$$H^2 = nh^2$$

as found before.